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# ON THE CONCEPTS OF NUMBER AND GROUP.

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There is a marked difference between the methods generally employed in the teaching of elementary geometry and arithmetic. In the former it is customary to begin by assuming that all the elements—points, lines, and planes—exist. By means of a sufficient number of postulates (axioms) these elements are then combined in such a way as to lead to a large number of interesting and valuable consequences. To complete the theory it is only necessary to prove that the postulates do not lead to any contradictory results and that they are sufficient to prove all geometric theorems.

In the theory of arithmetic we generally employ a widely different method. To develop the fundamental concept (number) we begin with the positive integers. This concept is gradually extended by the introduction of fractional, negative, irrational, and complex numbers. The last class may be regarded as pairs of real numbers. This method has been called the *genetic* method* because the general concept of number is reached by the gradual extension of the more elementary concept. The method described in the preceding paragraph may be distinguished as the *axiomatic* method.

While the genetic method seems to be the most suitable for the teaching of the elements of arithmetic it is doubtful whether it is so well suited for a logical presentation of the theory of this subject. At any rate it is a matter of great interest to develop the theory of arithmetic by the axiomatic method. The axioms of arithmetic have recently received considerable attention. The works of Dedekind, Burali-Forti, Padoa, Pieri, Peano, and others, along this line have lead to many interesting results. The last volume of th German Mathematical Association contains a paper by Professor Hilbert in which the concept of real numbers is presented according to the axiomatic method as follows :†

There is a system of elements (things) which are called numbers and they are represented by $a$, $b$, $c$, .... Their relations to each other and their laws of combination are completely defined by the following axioms :

## A. AXIOMS OF COMBINATION.

1. By adding $a$ and $b$ we obtain a definite number $c$, in symbols :

$$a+b=c \text{ or } c=a+b$$

2. If $a$ and $b$ are given there is always one and only one number $x$ and also one and only one number $y$ such that

$$a+x=b, \quad y+a=b$$

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*Hilbert, Jahresbericht der Deutschen Mathematik Vereinigung, 1900, page 180.

†Loc. cit.

3. There is a definite number, viz. 0, such that every number $a$ satisfies the equations

$$a+0=a \text{ and } 0+a=a$$

4. According to a second law of combination, called multiplication, we obtain a definite number $c$ from $a$ and $b$, in symbols

$$ab=c \text{ or } c=ab$$

5. When $a$ and $b$ are any given numbers, except $a$ cannot be 0, there is one and only one number $x$ and also one and only one number $y$ such that

$$ax=b, \quad ya=b$$

6. There is a definite number, viz. 1, such that every number $a$ satisfies the equations

$$a.1=a \text{ and } 1.a=a$$

## B. AXIOMS OF OPERATION.

The following formulas apply to any three numbers; $a$, $b$, $c$.

1. $a+(b+c)=(a+b)+c$.
2. $a+b=b+a$.
3. $a(bc)=(ab)c$.
4. $a(b+c)=ab+ac$.
5. $(a+b)c=ac+bc$.
6. $ab=ba$.

## C. AXIOMS OF ARRANGEMENT.

1. If $a$ and $b$ are two different numbers then one (say $a$) of them must be larger than the other ($b$). In this case $b$ is said to be smaller than $a$, in symbols $a>b$ and $b<a$.

2. From $a>b$ and $b>c$ it follows that $a>c$.

3. When $a>b$ the following inequalities must hold $a+c>b+c$ and $c+a>c+$b.

4. From $a>b$ and $c>0$ it follows that $ac>bc$ and $ca>cb$.

## D. AXIOMS OF CONTINUITY.

1. If $a$ and $b$ are any two numbers such that $a>0$ and $b>0$ it is always possible to add $a$ to itself a sufficient number of times so that the sum is greater than $b$; *i. e.*

$$a+a+\dots.+a>b$$

2. The numbers $a$, $b$, $c$, .... form a system which is completely defined by the preceding axioms ; *i. e.* it is impossible to find a different system of elements which obey all of these conditions.

Professor Hilbert observed that some of these axioms can be deduced from the others—in particular, it is easy to prove that the existence of 0 is a consequence of A 1, 2 and B 1, 2. From A 1, 2 it follows that for every number $a$ there is a number $x$ such that

$$a+x=a \text{ or } b+(a+x)=b+a$$

and that $a+b$ may represent any number while $a$ is fixed. From B 1 we observe that $b+(a+x)=(b+a)+x$ ; *i. e.* this number $x$ is independent of $a$. Finally, we conclude from B 2 that this number $x$ is the same in

$$\cdot a+x=a \text{ and } x+a=a$$

This proves that axiom A 3 is a consequence of A 1, 2 and B 1, 2.

Most of the thoughts expressed above may be found in the article by Professor Hilbert to which reference has been given. As the thoughts seem likely to interest teachers of mathematics and as the journal in which they appeared is probably inaccessible to most of the readers of this Journal it seemed desirable to reproduce them here with slight modifications. In his address before the Paris International Mathematical Congress, Professor Hilbert stated that one of the interesting mathematical problems which await solution is the determination of some one system of independent compatible axioms governing and defining arithmetic concepts. In the discussion of this paper Peano of Italy stated that several Italian mathematicians, viz. Burali-Forti, Padoa, and Pieri, had completely solved this problem in memoirs to which references are given in Vol. 7, of Rivista di Matematica.

It is interesting to observe that all the axioms of the group theory are included in the above axioms of arithmetic. The system of axioms (completely defining and governing the concepts of group theory) which is frequently given, is composed of A 4, 5 and B 3. When the order of the group is finite A 5 may be replaced by the following simpler axiom : From each of the equations

$$ab=ab' \text{ or } ba=b'a$$

it results that $b=b'$ It is thus apparent that the concept of operator in the theory of groups is much less restricted than the concept of real numbers, and includes the latter with respect to multiplication. Numbers have, however, an additional law of combination, viz. addition, while the operators of a group are combined according to a single law of combination generally called multiplication.